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THE STEADY ANTIPLANE DYNAMIC CONTACT PROBLEM OF A PERIODIC STRUCTURE FOR AN ELASTIC HALF-SPACE†

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The antiplane steady dynamic contact problem for a periodic system of punches in an elastic half-space, which is analogous to the plane problem which has been treated earlier [1], is studied. Considerable attention is paid to investigating the resonance phenomena which arise when harmonic loads act on the punches. © 1996 Elsevier Science Ltd. All rights reserved.

1. The antiplane time-independent motion of a homogeneous isotropic elastic half-space ($y \geqslant 0$) is considered (the y axis assumed to be directed downwards) and a shear load $\tau_{zy}/(2\mu) = f(x)\exp(-i\omega t)$ (μ is Lamé's constant) is applied at the upper edge of the axis ($y = 0$). Motion of the half-space for which the displacements of the points are parallel to the z axis is understood as an antiplane motion.

The amplitude displacements of the points of the half-space are determined from the formula [2]

$$w(\omega, x, y) = -\frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi)\gamma^{-1} \exp(i\xi x - \gamma y) d\xi \tag{1.1}$$

$$f(\xi) = \int_{-\infty}^{\infty} f(x) \exp(-i\xi x) dx, \quad \gamma = (\xi^2 - \kappa^2)^{1/2}, \quad \kappa = \frac{\omega}{\upsilon}$$

Here, $f(\xi)$ is the Fourier transform of the function $f(x)$, υ is the propagation velocity of transverse waves in the elastic medium and, starting from the radiation conditions, it is assumed that $\gamma \geqslant 0$ when $|\xi| \geqslant \kappa$ and $\gamma = -i(\kappa^2 - \xi^2)^{1/2}$ when $|\xi| < \kappa$.

We shall now assume that the load on the surface of the half-space is quasi-periodic, that is, we assume that

$$f(x + l) = f(x)\exp(-i\alpha) \tag{1.2}$$

where l is the length of a certain segment and α is a real parameter, $|\alpha| \leqslant \pi$.

In this case, it can be shown that

$$f(\xi) = g(\xi) \sum_m \exp[-im(\xi l + \alpha)], \quad g(\xi) = \int_{-l/2}^{l/2} f(x) \exp(-i\xi x) dx \tag{1.3}$$

The convergence of the series in (1.3) is to be understood in the sense of convergence in the space of generalized functions [3].

Substituting (1.3) into (1.1) and using the equality [3]

$$\sum_m \exp(iml\xi) = \frac{2\pi}{l} \sum_k \delta\left(\xi - k\frac{2\pi}{l}\right)$$

(where $\delta(\xi)$ is the δ-function), we find that

$$w(\omega, x, y) = -\frac{2}{l} \sum_k g_k \gamma_k^{-1} \exp(-\gamma_k y + i\xi_k x) \tag{1.4}$$

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$$\xi_k = (2k\pi - \alpha)/l, \quad g_k = g(\xi_k), \quad \gamma_k = (\xi_k^2 - \kappa^2)^{1/2}$$

By assuming that the quasi-periodic load consists of concentrated loads, that is, $f(x) = -\delta(x)$ when $|x| < l/2$, we obtain from (1.4), when $y = 0$, the so-called group function of the effect

$$K(\alpha, \omega, x) = \frac{2}{l} \sum_k \gamma_k^{-1} \exp(i\xi_k x) \tag{1.5}$$

since, in this case, $g(\xi) = 1$.

When $\gamma_k = 0$, the group function of the effect becomes infinite. It is easily explained that this occurs at frequencies $\omega_k = |2k\pi - \alpha| \upsilon/l$ ($k \in Z$, where Z is the set of real integers including zero). These frequencies correspond to the cut-off frequencies of a layer $|x| \leq l/2$ with the following boundary conditions

$$w(l/2, y) = w(-l/2, y)\exp(-i\alpha), \quad \tau_{zx}(l/2, y) = \tau_{zx}(-l/2, y)\exp(-i\alpha)$$

Note that the quasi-periodic problem for a half-space (that is, when there is a quasi-periodic load on the surface) is equivalent to the problem for the half-layer $|x| \leq l/2, y \geq 0$ with the above-mentioned boundary conditions. The phenomenon of infinitely large displacements can be explained by the fact that the energy from a source located on the surface can not progress within the half-layer since it is "cut-off" for waves which are travelling downwards. We shall call these frequencies the α-resonance frequencies for the half-space.

We now consider the motion of the half-space in the case of a kinetic perturbation, that is, the motion caused by the movement of strip punches with a plane base, arranged on the boundary of the half-space, for displacements of the punches of specified amplitude (Fig. 1). We shall henceforth refer to this problem as the kinematic problem. The punches are parallel to the z axis, their width is equal to $2a$ and the distance between their axes is equal to l. The punches are successively numbered from $-\infty$ to $+\infty$. The quasi-periodic problem is studied, in which the amplitude displacements of the punches are specified in the following manner

$$w_m = \exp(-im\alpha) \tag{1.6}$$

The contact stresses will then also possess the property of quasi-periodicity (1.2). It follows from this that it is sufficient to determine the contact stresses under one of the punches, under the "main" punch, for example, which has the number zero. On equating the displacements of the points of the boundary of the half-space under the main punch to the displacement of the punch, we obtain the integral equation of the problem

$$\int_{-a}^{a} K(\alpha, \omega, x - s)p(s)ds = 1 \quad (|x| \leq a) \tag{1.7}$$

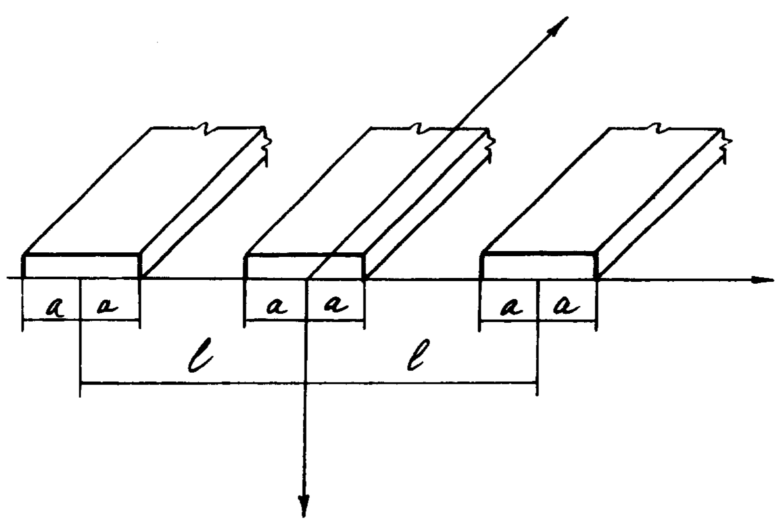

Fig. 1.

Here, $p(s)$ are the contact stresses.

We will first consider the case when the frequency of the motion of the punches is different from all the α-resonance for the half-space. Then, by representing γ_k in the form

$$\gamma_k = |\xi_k|[1-(\kappa/\xi_k)^2]^{1/2}$$

we obtain the asymptotic representation of γ_k^{-1} for large $|k|$

$$\gamma_k^{-1} = \frac{l}{2\pi|k|}\left(1+\frac{\alpha}{2\pi k}\right)+r_k, \quad r_k = O(|k|^{-3}) \tag{1.8}$$

From (1.5) and (1.8), we obtain

$$K(\alpha,\omega,x) = \frac{1}{\pi}\exp\left(-i\frac{\alpha}{l}x\right)\left[\sum_{k=1}^{\infty} k^{-1}\cos\left(\frac{2\pi}{l}kx\right)+\right.$$
$$\left.+i\frac{\alpha}{2\pi}\sum_{k=1}^{\infty} k^{-2}\sin\left(\frac{2\pi}{l}kx\right)\right]+K_1(\alpha,\omega,x) \tag{1.9}$$
$$K_1(\alpha,\omega,x) = \frac{2}{l}\exp\left(-i\frac{\alpha}{l}x\right)\sum_{k\neq 0} r_k \exp\left(i\frac{2\pi}{l}kx\right)-\frac{1}{\pi\gamma_0}$$

Using the well-known formula [4]

$$\frac{1}{2}\ln[2(1-\cos x)] = -\sum_{k=1}^{\infty} k^{-1}\cos(kx)$$

we find that

$$K(\alpha,\omega,x) = \frac{1}{\pi}\exp\left(-i\frac{\alpha}{l}x\right)\left[\ln\frac{a}{|x|}+i\frac{\alpha}{2\pi}\sum_{k=1}^{\infty} k^{-2}\sin\left(\frac{2\pi}{l}kx\right)\right]+K_2(\alpha,\omega,x)$$
$$K_2(\alpha,\omega,x) = \frac{1}{\pi}\exp\left(-i\frac{\alpha}{l}x\right)\ln\left\{|x|\left[2a\left|\sin\left(\frac{\pi}{l}x\right)\right|\right]^{-1}\right\}+K_1(\alpha,\omega,x) \tag{1.10}$$

It can be shown that $K_2(\alpha, \omega, x)$ are doubly continuously differentiable functions of x in the range $|x| \leqslant a$ $(a < l)$.

It follows from (1.10) that the kernel $K(\alpha, \omega, x)$ is integrable with a square in the range $|x| \leqslant a$ and, consequently, Eq. (1.7) is a Fredholm integral equation of the first kind and the term $\pi^{-1}\exp(-i\alpha/lx)\ln|x|^{-1}$ represents the so-called "singular" part of the kernel.

It follows from the relation

$$\int_{-1}^{1} \ln|x-y|T_j(y)[1-y^2]^{-1/2}dy = -\nu_j T_j(x) \quad (j=0,1,2,\ldots)$$

($\nu_0 = \pi\ln 2$, $\nu_j = \pi/j$ when $j > 0$) that the eigenfunctions of the singular part of the kernel are Chebyshev polynomials $T_j(y)$. The solution of Eq. (1.7) will therefore be sought in the form

$$p(s) = \sum_{j=0}^{\infty} p_j T_j\left(\frac{s}{a}\right)\left[1-\left(\frac{s}{a}\right)^2\right]^{-1/2} \tag{1.11}$$

Substituting (1.11) into (1.7), multiplying by $T_n(x/a)[1-(x/a)^2]^{-1/2}$ and integrating with respect to x from $-a$ to $+a$, we arrive at an infinite system of linear algebraic equations.

$$\sum_{j=0}^{\infty} A_{nj} p_j = l\delta_{n0} \tag{1.12}$$

$$A_{nj} = 2a^2\pi i^{j-n} \sum_k \gamma_k^{-1} J_n(\xi_k a) J_j(\xi_k a)$$

where δ_{nm} is the Kronecker delta and $J_n(x)$ is a Bessel function.

Using representation (1.10), it can be shown that system (1.12) is quasi-regular and is solved by reduction.

We will now assume that the values of ω are close to an α-resonance frequency for the half-space ω_k (the subscript k is fixed). Then, γ_k may be as close to zero as desired and the kth term in (1.5) can take values as large as desired. On multiplying (1.7) by γ_k, we shall have

$$V_1 p + \varepsilon V_2 p = \varepsilon, \quad \varepsilon = \gamma_k \tag{1.13}$$

$$V_1 p = -\frac{2}{l} \exp(i\xi_k x) \int_{-a}^{a} \exp(-i\xi_k s) p(s) ds$$

$$V_2 p = -\frac{2}{l} \int_{-a}^{a} \sum_{m \neq k} \gamma_m^{-1} \exp[i\xi_m (x - s)] p(s) ds$$

Note that V_1 is a finite dimensional operator which maps the space $L(-a, a)$ into the one-dimensional subspace spanned by the vector $\exp(i\xi_k x)$.

We shall seek a solution of Eq. (1.13) in the following form

$$p(y) = \sum_{m=0}^{\infty} q_m h_m(y) [1 - (y/a)^2]^{-1/2} \tag{1.14}$$

$$h_0(y) = \exp(i\xi_k y), \quad h_m(y) = T_{m-1}(y) - d_m \exp(i\xi_k y) \quad (m > 0)$$

The coefficients d_m are determined from the orthogonality condition

$$\int_{-a}^{a} h_m(y) [1 - (y/a)^2]^{-1/2} \exp(-i\xi_k y) dy = 0 \quad (m > 0)$$

It is easy to determine that $d_m = i^{m-1} \pi J_{m-1}(\xi_k a)$.

We substitute (1.14) into (1.13), multiply by $h_n(x)$ ($n = 0, 1, 2, \ldots$) and integrate with respect to x from $-a$ to $+a$. As a result, we shall have

$$\frac{2}{l} a^2 \pi^2 q_0 + \varepsilon \sum_{m=0}^{\infty} q_m (V_2 h_m, h_0) = \varepsilon (1, h_0)$$

$$\sum_{m=0}^{\infty} q_m (V_2 h_m, h_n) = (1, h_n) \quad (n > 0) \tag{1.15}$$

$$(f_1, f_2) = \int_{-a}^{a} f_1(x) \bar{f}_2(x) dx$$

Eliminating q_0 from (1.15), we arrive at the following system of linear algebraic equations for q_m ($m > 0$)

$$\sum_{m=1}^{\infty} B_{nm} q_m = b_n \quad (n = 1, 2, \ldots) \tag{1.16}$$

$$B_{nm} = (V_2 h_m, h_n) - \frac{\varepsilon l (V_2 h_m, h_0)(V_2 h_0, h_n)}{2a^2\pi^2 + \varepsilon l (V_2 h_0, h_0)}$$

$$b_n = (1, h_n) - \frac{\varepsilon l (1, h_0)(V_2 h_0, h_n)}{2a^2\pi^2 + \varepsilon l (V_2 h_0, h_0)}$$

It can be shown that

$$(V_2 h_m, h_n) = 2a^2\pi^2 \sum_{j\neq k} \gamma_j^{-1}[(-i)^{n-1} J_{n-1}(\xi_j a) - \bar{d}_n J_0(\xi_k a - \xi_j a)]\times$$
$$\times[i^{m-1} J_{m-1}(\xi_j a) - d_m J_0(\xi_k a - \xi_j a)] - \tag{1.17}$$
$$-[\delta_{n1} - \bar{d}_n J_0(\xi_k a)][\delta_{m-1} - d_m J_0(\xi_k a)]$$

It follows from (1.17) that system (1.16) is quasi-regular for any sufficiently small ε (excluding $\varepsilon = 0$).

2. We shall call those frequencies, at which the principal vector of the contact stresses under a punch becomes infinite, the α = resonance frequencies of the quasi-periodic kinematic problem. It is obvious that the existence of such frequencies is associated with the occurrence of non-trivial solutions in the case of the equation

$$\int_{-a}^{a} K(\alpha, \omega, x-s) p(s) ds = 0 \quad (|x| \leqslant a) \tag{2.1}$$

Substituting (1.5) into (2.1), we obtain

$$\sum_k p_k \gamma_k^{-1} \psi_k(x) = 0 \quad (|x| \leqslant a) \tag{2.2}$$

$$\psi_k(x) = l^{-1/2} \exp(i\xi_k x), \qquad p_k = \int_{-a}^{a} p(y) \overline{\psi}_k(y) dy$$

We shall investigate the question of the occurrence of non-trivial solutions of (2.1) in $L(-a, a)$ and assume that they belong to $L(-a, l-a)$ and vanish almost everywhere in $[a, l-a)$, that is, they satisfy the equalities

$$\int_{a}^{l-a} p(s) \overline{\varphi}_n(s) ds = 0 \quad (n \in Z) \tag{2.3}$$

where $\{\varphi_n(y)\}_{-\infty}^{\infty}$ is a basis of the Banach space $C(a, l-a)$.

Let $\{\eta_n(x)\}_{-\infty}^{\infty}$ be a basis of the space $C(-a, a)$. Multiplying (2.2) by $\eta_{-n}(x)$ and integrating from $-a$ to a, we obtain

$$\sum_k p_k \gamma_k^{-1} (\psi_k, \eta_n) = 0 \quad (\eta_n(x) \equiv 0, \quad x \in (a, l-a]) \tag{2.4}$$

Here and below

$$(f_1, f_2) = \int_{-a}^{l-a} f_1(x) \bar{f}_2(x) dx$$

Substituting the expansion

$$p(y) = \sum_k p_k \psi_k(x)$$

into (2.3), we shall have

$$\sum_k p_k (\psi_k, \varphi_n) = 0 \quad (\varphi_n(x) \equiv 0, \quad x \in [-a, a)) \tag{2.5}$$

Introducing the notation $x_k = p_k/\gamma_k$, $b_{nk} = (\psi_k, \varphi_n)$, $c_{nk} = (\psi_k, \eta_n)$, we write (2.4) and (2.5) in the following form

$$\sum_k x_k \gamma_k b_{nk} = 0, \quad \sum_k x_k c_{nk} = 0 \quad (n \in Z) \tag{2.6}$$

The functions $\{\psi_k(x)\}_{-\infty}^{\infty}$ form an orthonormal basis of the Hilbert space $H = L_2(-a, l-a)$ and

$\{\eta_k(x)\}_{-\infty}^{\infty}$ and $\{\varphi_k(x)\}_{-\infty}^{\infty}$ are bases of the spaces $H_1 = L_2(-a, a)$ and $H = L_2(-a, l-a)$, respectively. It is obvious that $H = H_1 \oplus H_2$. It may be assumed without any loss of generality that $\{\eta_n(x)\}_{-\infty}^{\infty}$ is an orthonormalized basis in H_1.

We will now investigate the question of the existence of non-trivial solutions in the case of system (2.6) in l_2.

We note that systems which are similar to (2.6) in a certain sense have been investigated previously in [5]. However, there are considerable differences between them: the systems considered in [5] refer to systems of the second kind, and (2.6) is of the first kind. Furthermore, the simplifying assumption that all $\gamma_k \neq 0$ was adopted, whereas a finite number of γ_k vanish here in the case of the α-resonance frequencies. All of this has a considerable effect on the choice of methods of investigation. The results and, to a large extent, the methods used in [5] cannot be applied to (2.6).

It can be shown that γ_k satisfy the following conditions:
1. each of them is either a real or a purely imaginary number and all of the non-zero real numbers are of the same sign (positive);
2. the number of γ_k which are purely imaginary and equal to zero is finite.

Lemma. If

$$\{\gamma_k b_{mk}\}_{k=-\infty}^{\infty} \in l_2 \tag{2.7}$$

for any m, then system (2.6) only has a trivial solution in l_2.

Proof. We introduce the operators **G** and **G*** using the formulae

$$Gf = \sum_k \gamma_k (f, \psi_k)\psi_k, \quad G^* f = \sum_k \gamma_k (f, \psi_k)\psi_k$$

which are defined in a certain set E_G, which is dense everywhere in H and consists of the elements $f \in H$ for which the series $\Sigma_k |\gamma_k (f, \psi_k)|^2$ converges. Condition (2.7) means that $\varphi_m \in E_G$.

Definition. A linear operator T, defined in a certain set E_G of Hilbert space belongs to class (S) if it follows from $(Tf_n, f_n) \to 0$ that $f_n \to 0$.

Let us show that $G \in (S)$ in H_2. Let $f \in E_G$. Using the notation $(f, \psi_k) = f_k$, we find

$$(Gf, f) = G_1(f) + iG_2(f) \tag{2.8}$$

$$G_1(f) = \sum_k \operatorname{Re}\gamma_k |f_k|^2, \quad G_2(f) = \sum_k \operatorname{Im}\gamma_k |f_k|^2$$

We note that, in the sum which defines $G_1(f)$, a finite number of terms are identically equal to zero. We shall show that $G_1(f)$ is positive definite in H_2. Let us assume the opposite, that is, that a sequence of elements $f_n \in H_2(\|f_n\| = 1, n = 1, 2, \ldots)$ exists which are such that $G_1(f_n) \to 0$. We will represent an element f_n in the following form: $f_n = f_n^{(0)} + f_n^{(1)}, f_n^{(0)} \in H_0$ (H_0 is the linear hull of the elements ψ_p for which $\operatorname{Re}\gamma_p = 0$ and $f_n^{(1)} \perp f_n^{(0)}$). Then, $G_1(f_n) = G_1(f_n^{(1)}$ and $G_1(f_n^{(1)}) \geqslant \gamma\| (f_n^{(1)}) \|$, where $\gamma = \min \gamma_k$ (for all k to which $\operatorname{Re}\gamma_k \neq 0$ corresponds). Hence $\|f_n^{(1)}\| \to 0$ when $n \to \infty$, that is, $\|f_n - f_n^{(0)}\| \to 0$.

It follows from this that all of the $\|f_n^{(0)}\|$ $(n = 1, 2, \ldots)$ are bounded by a single number. It follows from the finite dimensionality of H_0 that a converging subsequence $f_q^{(0)} \to f^{(0)} \in H_0(q = n_1, n_2, \ldots)$ can be chosen from the sequence $f_n^{(0)}$ $(n = 1, 2, \ldots)$. Then, f_q $(q = n_1, n_2, \ldots)$ also tends to this limit. Since H_2 is complete, $f^{(0)} \in H_2$. Hence, $f^{(0)} \in H_0 \cap H_2$. But $H_0 \cap H_2 = \{0\}$, since the shortest distance from any vector $\psi_k = l^{-1/2}\exp(i\xi_k x)$ to H_2 is not less than the quantity

$$s = \left[\int_a^{l-a} dx\right]^{1/2} = (l-2a)^{1/2} \tag{2.9}$$

A contradiction has been arrived at: one the one hand, $\|f^{(0)}\| = 1$ as the limit of $\|f_q\|$ and, on the other hand, $f^{(0)} = 0$, and so $G_1(f) > \rho > 0$ in H_2.

Let us now assume that $(Gf_m, f_m) \to 0$ for a certain sequence of elements $f_m \in H_2$. It follows from (2.8) that $G_1(f_m) \to 0$ and this is only possible for $f_m \to 0$. Hence $G \in (S)$ in H_2. It can similarly be proved that $G^* \in (S)$ in H_2.

Using the operator G, system (2.6) can be written in the following form

$$(Gx,\varphi_m)=0,\quad (x,\eta_m)=0\quad (m\in Z) \tag{2.10}$$

Let system (2.10) have a non-zero solution $x \in E_G$. It follows from the second equality of (2.10) that $x \in H_2$ and, from the first equality, that $Gx \in H_1$. Hence $(G_x, x) = 0$, and it follows from the fact that G belongs to the class (S) that $x = 0$.

If, however, $x \in E_G$, then a sequence of elements $x_k \to x$ can be found in $H_2 \cap E_G$ since E_G is everywhere dense in H. We shall write (2.10) as

$$(x,G^*\varphi_m)=0,\quad (x,\eta_m)=0\quad (m\in Z)$$

It then follows from the fact that x_k tends to x that $\lim_{k\to\infty}(x_k, G^*\varphi_m) = 0$ for each m. Hence

$$\lim_{k\to\infty}(Gx_k,\varphi_m)=0,\quad (x_k,\eta_m)=0\quad (m\in Z)$$

We now introduce the notation $G_{km} = (Gx_k, \varphi_m)$. It may be assumed without loss of generality that $|G_{km}|$ decrease monotonically for a fixed m. Then, the series $\Sigma_m |G_{km}|^2 \to 0$ when $k \to \infty$.

Actually, for any ε which may be as small as desired, it is possible to obtain an m_0 such that

$$\sum_{m=m_0+1}^{\infty}(|G_{1m}|^2+|G_{1,-m}|^2)<\varepsilon/2$$

On the other hand, a k_0 can be found such that

$$\sum_{m=-m_0}^{m_0}|G_{km}|^2<\varepsilon/2$$

when $k > k_0$ and, consequently, for such k

$$\sum_m |G_{km}|^2<\sum_{m=-m_0}^{m_0}|G_{km}|^2+\sum_{m=m_0+1}^{\infty}(|G_{1m}|^2+|G_{1,-m}|^2)<\varepsilon$$

which also means that the above-mentioned series tends to zero.

But

$$(Gx_k,x_k)=\left(Gx_k,\sum_m x_{km}\varphi_m\right)=\sum_m x_{km}G_{km}$$

and, using the Cauchy–Bunyakovskii inequality, we obtain

$$|(Gx_k,x_k)|\leqslant\left[\sum_m |x_{km}|^2\right]^{1/2}\left[\sum_m |G_{km}|^2\right]^{1/2}$$

When $k \to \infty$, the first factor on the right-hand side of the inequality tends to $\|x\|$ and the second tends to $(Gx_k, x_k) \to 0$, and it follows from the fact that G belongs to class (S) that $x = \lim_{k\to\infty} x_k = 0$.

Hence the lemma is proved.

Theorem. For any $\omega > 0$, the integral equation (2.1) only has a trivial solution in $L(-a, a)$.

Proof. We construct a sequence of functions $\varphi_m(x)(m \in Z)$ in H_2 as follows:

$$\varphi_m(x)=\varphi_m^0(x)\zeta_m(x) \tag{2.11}$$

$$\varphi_m^0(x)=\exp[i2\pi m(x-a)/b]/\sqrt{b},\quad b=l-2a\quad (a\leqslant x\leqslant l-a)$$

$$\zeta_m(x)=\begin{cases}(x-a)/\varepsilon_m & (a\leqslant x\leqslant a+\varepsilon_m)\\ 1 & (a+\varepsilon_m\leqslant x\leqslant l-a-\varepsilon_m)\\ (l-a-x)/\varepsilon_m & (l-a-\varepsilon_m\leqslant x\leqslant l-a)\end{cases} \tag{2.12}$$

Here, $\varepsilon_m > 0$ is a certain summable sequence $(\Sigma_m\varepsilon_m < \infty)$.

For example, it is possible to put $\varepsilon_m = \varepsilon_0/(|m| + 1)^2$. We note that $|\zeta_m(x)| \leqslant 1$ and $\varphi_m^0(x)$ is an orthonormalized sequence.

It can be shown that (2.11) is a basis which is quadratically close to an orthonormalized basis [6]. For this purpose, it is sufficient that the operator T, defined by the formulae

$$T\varphi_m^0 = \varphi_m - \varphi_m^0 (m \in Z)$$

satisfies the following conditions [6]:

(a) the operator T is a Hilbert–Schmidt operator;

(b) the operator $I + T$ is invertible (I is a unit operator in the space H_2).

It follows from (2.12) that

$$\|\varphi_m - \varphi_m^0\|^2 = \int_a^{l-a} |\varphi_m(x) - \varphi_m^0(x)|^2\, dx \leqslant \frac{1}{b} \int_a^{l-a} |\zeta_m(x) - 1|^2\, dx \leqslant \frac{2\varepsilon_m}{b}$$

Then

$$\sum_m \|T\varphi_m^0\|^2 = \sum_m \|\varphi_m - \varphi_m^0\|^2 \leqslant \frac{2}{b} \sum_m \varepsilon_m < \infty$$

This also means that T is a Hilbert–Schmidt operator [6].

In order to check the δ condition we shall take a sequence of complex numbers z_k ($k \in Z$) such that $\Sigma_k |z_k|^2 \leqslant 1$. An element $\varphi(x) = \Sigma_k z_k \varphi_k^0(x)$ of the space H_2 has a norm which does not exceed unity. Then

$$\|T\varphi\| \leqslant \sum_k |z_k|\, \|T\varphi_k^0\| = \sum_k |z_k|\, \|\varphi_k - \varphi_k^0\|$$

Using the Cauchy–Bunyakovskii inequality, we find

$$\|T\varphi\| \leqslant \left[\sum_k |z_k|^2\right]^{1/2} \left[\sum_k \|\varphi_k - \varphi_k^0\|^2\right]^{1/2} \leqslant \left[\frac{2}{b} \sum_m \varepsilon_m\right]^{1/2}$$

If the magnitude of ε_m is chosen such that $\Sigma_m \varepsilon_m < b/2$, it then turns out that $\| T_\varphi \| < 1$ and this also means that the inequality $\| T_\varphi \| < 1$ is satisfied. It follows from this that the operator $I + T$ is invertible, that is, that the δ condition is satisfied. Hence, the sequence $\varphi_m(x)$ ($m \in Z$) is a basis of the space H_2.

On integrating by parts, we find

$$b_{mk} = \frac{i}{\sqrt{l}\xi_k} \int_a^{l-a} \varphi_m(y) \exp(i\xi_k y) dy \tag{2.13}$$

Using (2.11) and (2.12), we find

$$b_{mk} = \frac{2\pi m}{b\xi_k} b_{mk} - \frac{d_{mk}}{\sqrt{b - l}\xi_k \varepsilon_m (2\pi m / b - \xi_k)}$$

$$d_{mk} = \exp[-i\varepsilon_m 2\pi m / b + i\xi_k (a + \varepsilon_m)] - \exp(i\xi_k a) -$$
$$- \exp[i\xi_k (l - a)] + \exp[i\varepsilon_m 2\pi m / b + i\xi_k (l - a - \varepsilon_m)]$$

It follows from this that

$$b_{mk} = d_{mk} \left[\sqrt{bl} \varepsilon_m (2\pi m / b - \xi_k)^2\right]^{-1} \tag{2.14}$$

Since $|d_{mk}| \leqslant 4$ and $\gamma_k = O(|k|)$, it follows from (2.14) that $\{\gamma_k b_{mk}\}_{k=-\infty}^{\infty} \in l_2$ for any m. Hence, the conditions of the lemma are satisfied and system (2.6) only has a trivial solution in l_2.

Since $x_k \in l_2$, then $x_k = o(|k|^{-1/2})$ and we obtain from the equality $p_k = x_k \gamma_k$ and the asymptotic form $\gamma_k \sim 2\pi |k| / l$ that $p_k = o(|k|^{1/2})$. It follows from this that the class of functions in which Eq. (2.1) only has a trivial solution contains $L(-a, a)$.

The theorem is proved.

It follows from the theorem that Eq. (1.7) has a unique solution in $L(-a, a)$. Hence, the quasi-periodic kinematic problem does not have α-resonance frequencies, that is, there are no travelling waves in the half-space $y \geqslant 0$, which is fixed at the upper bound along the bands $|x - lm| \leqslant a$ $(m \in Z)$.

The numerical investigation enables one to clarify the physical reason for the boundedness of the principal vector of the contact stresses for any α and ω. We shall compare the field of the directions of the power flux under the punch, averaged over a period of the flux, at the non- α-resonance value of the frequency $\omega = 0.8$ (Fig. 2) and at the α-resonance value of the frequency for the half-space $\omega_1 = 1.571$ (Fig. 3). At the non-resonance value ω, the power flux goes to infinity while, at the resonance value of the half-layer $|x| \leqslant l/2$, the power flux is "cut-off" and a balance between the input and output power fluxes under the punch is observed. In the example under consideration $\alpha = 0, l = 4$ (ω and l are dimensionless: $\omega = \omega' a/\upsilon, l = l'/a$; ω' and l' are the dimensional quantities).

3. We will now consider the quasi-periodic contact problem when the displacements of the punches are not specified but the loads acting on them are given. The amplitude values of the loads are quasi-periodic, that is, $F_m(\alpha) = F_0(\alpha)\exp(-im\alpha)$ (m is the number of the punch and $m \in Z$).

We write down the differential equation of the motion of the main punch

$$\mu_0 = d^2 w_0(t)/dt^2 = F_0(\alpha)\exp(-i\omega t) - R_0(t) \tag{3.1}$$

Here, μ_0 is the mass per unit length of the punch, $w_0(t)$ is its displacement and $R_0(t)$ is the resultant of the contact stresses.

On looking for the steady motion, we arrive at the equality

$$-\mu_0\omega^2 w_0(\alpha, \omega) = F_0(\alpha) - R_0(\alpha, \omega) \tag{3.2}$$

where $w_0(\alpha, \omega)$ and $R_0(\alpha, \omega)$ are the amplitude values of $w_0(t)$ and $R_0(t)$.

On equating the displacements of the punch to the displacements of points of the boundary of the half-space, we obtain the integral equation

$$\int_{-a}^{a} K(\alpha, \omega, x - s) q_0(\alpha, \omega, s) ds = -\frac{F_0(\alpha) - R_0(\alpha, \omega)}{\mu_0\omega^2} \quad (|x| \leqslant a) \tag{3.3}$$

Here, $q_0(\alpha, \omega)$ are the contact stresses under the main punch.

Comparing Eqs (1.7) and (3.3), we find

$$R_0(\alpha, \omega) = F_0(\alpha) P_0(\alpha, \omega) / [P_0(\alpha, \omega) - \mu_0\omega^2] \tag{3.4}$$

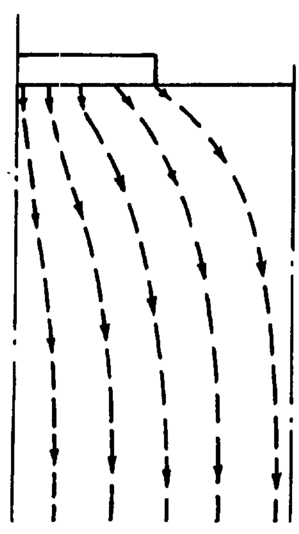

Fig. 2.

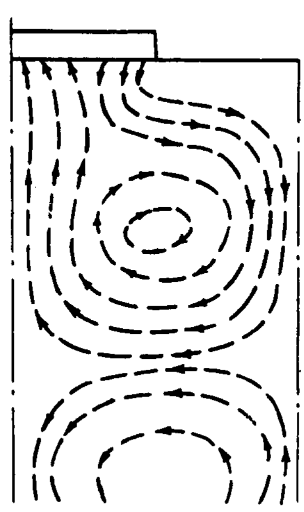

Fig. 3.

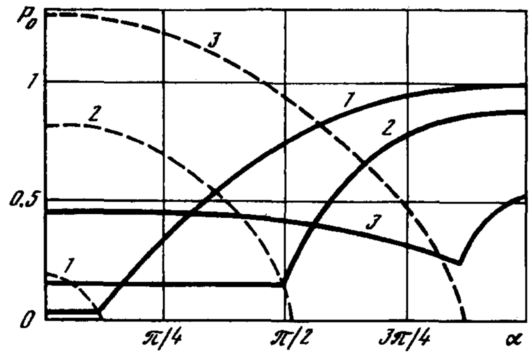

Fig. 4.

Here, $P_0(\alpha, \omega)$ is the principal vector of the contact stresses under the punch in the kinematic contact problem.

Hence, in spite of the fact that the quasi-periodic kinematic problem does not have α-resonances, the analogous contact problem can have them if $P_0(\alpha, \omega)$ takes real positive values.

The graphs of $P_0(\alpha, \omega)$ against α shown in Fig. 4 for $l = 4$ and $\omega = 0.1$ (curve 1), $\omega = 0.4$ (curve 2) and $\omega = 0.7$ (curve 3) (the solid line corresponds to Re $P_0(\alpha, \omega)$ and the dashed line to Im $P_0(\alpha, \omega)$) show that values of α and ω exist for which $P_0(\alpha, \omega) > 0$. It can be shown that the domain of these values is determined by the inequality $\omega \leqslant \upsilon \mid \alpha \mid /l$

We shall now consider the general contact problem for a periodic system of punches when the presence in the load of any symmetry properties whatsoever is not assumed. The resultant of the contact stresses under the mth punch is given by the formula [1]

$$R_m(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_0(\alpha, \omega) \exp(-im\alpha) d\alpha = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F_0(\alpha) P_0(\alpha, \omega) e^{-im\alpha}}{P_0(\alpha, \omega) - \mu_0 \omega^2} d\alpha \tag{3.5}$$

Hence, the problem of the existence of resonances in the case of the general contact problem reduces to the question of the existence of the integral in (3.5), which is understood in the senses of the principal Cauchy value. It does not exist if the denominator of the integrand in (3.5) has a pole of second or higher order in the range $[-\pi, \pi]$, that is, if $\mu_0\omega^2 = P_0(\alpha, \omega)$ and $\partial P_0(\alpha, \omega)/\partial\alpha = 0$.

We note that the integral operator (1.7) is analytic outside the circle $\mid \alpha \mid \leqslant \omega l/\upsilon$ in the complex plane of α and this means [7] that the function $P_0(\alpha, \omega)$ is analytic with respect to α in this domain. Consequently, $(\omega l/\upsilon, \pi)$ and $\partial P_0(\alpha, \omega)/\partial\alpha$ depend continuously on α when $\omega l/\upsilon < \mid \alpha \mid \leqslant \pi$.

It is seen in Fig. 4 that, in the interval $(\omega l/\upsilon, \pi)$, the function $P_0(\alpha, \omega)$ decreases monotonically as α increases. Furthermore, $P_0(\alpha, \omega)$ is symmetrical about the axes $\alpha = k\pi (k \in Z)$ and periodic with period 2π. It follows from this that $\partial P_0(\pi, \omega)/\partial\alpha = 0$.

Hence, if $\mu_0\omega^2 = P_0(\pi, \omega)$ for $\omega \leqslant \upsilon\pi/l$, then $R_m(\omega)$ takes infinitely large values, that is, resonance sets in. As the frequency approaches the resonance value, the motion of the punch becomes antiperiodic, that is, the phases of the oscillations of two adjacent punches become opposite with a simultaneous increase in the amplitude of the oscillations.

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